Johns Hopkins University



Entropy methods for Porous Medium Equation

Research Report

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Summary

From February to August 2025, I was a Visiting Graduate Student in the Department of Mathematics at Johns Hopkins University (Baltimore, MD), where I carried out a six-month research internship under the supervision of Professor Sire.

The work focused on the study of the Fractional Porous Medium Equation. The solutions to this equation converge to a stationary solution over time. My goal was to prove, under suitable assumptions, that this convergence occurs at an exponential rate.

This document is to present a overview of the research I did during my stay at Hopkins.

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Chapter 1

Presentation

1.1 The Problem

The goal is to further understand the asymptotic behavior of the following equation's solution:

$$\begin{cases} \partial_t v = -\mathcal{L}\varphi(v) + cv & in\ \mathbb{R}_+^* \times \Omega, \\ v(0,x) = u_0(x) & in\ \Omega, \\ v(t,x) = 0 & in\ \mathbb{R}_+^* \times\ \partial\Omega \end{cases} \tag{3.1}$$

Where the linear operator $\mathcal{L}$ is a fractional power of the Laplacian: $\mathcal{L}(.) = (-\Delta)^s(.)$ with $s \in]0,1]$. φ is a smooth increasing function whose caracteritics will be built later, $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary, $d \geq 1$ and $c > 0$.

We call S a stationary solution of the equation, which means:

$$\begin{cases} \mathcal{L}\varphi(S) = cS & in\ \Omega, \\ \partial_t S = 0 & in\ \Omega, \end{cases} \tag{3.2}$$

Assuming the convergence of $v(t,x)$ in its stationary solution $S(x)$ as $t \to +\infty$ uniformly in space, we want to study the quantitative aspect of that convergence (under necessary assumptions), namely we want to obtain a rate of convergence of v to S with an estimate like

$$\|v(t,.) - S\|_X \leq\ a\, e^{-bt}$$

where $a,\ b$ depends only on the initial data (and parameters of the problem) and X is a suitable Hilbert space.

1.2 Qualitative context of the problem

In this section, we give some qualitative insight to better understand the origin of the problem exposed in the previous section.

First, let's give a definition of the fractionnal Laplacian. The fractional Laplacian is a generalization of the classical Laplacian. While the classical Laplacian Δu operates locally, measuring the sum of variations in u in all directions, the fractional Laplacian introduces a non-local component, capturing more complex diffusive effects.

For a fractional order $s \in]0,1[$, the fractional Laplacian of u, denoted by $(-\Delta)^s u$, can be defined in several equivalent ways (see [1]), with the most common being an integral formula:

$$\mathcal{L}u = (-\Delta)^s u(x) = C_{d,s}\,\text{P.V.} \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2s}}\,dy,$$

where $C_{n,s}$ is a normalization constant that depends on the dimension n and order s, and P.V. is the principal value.

The problem we study initially comes from this equation:

$$\begin{cases} u_t + \mathcal{L}(\varphi(u)) = 0 & \text{in } (0,+\infty)\times\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases}$$

where φ is a real function we will specify later. The leading example will be $\varphi(u) = |u|^{m-1}u$ with $m > 0$. We know and it is admitted that there exist a unique solution to this problem in a certain class of solution (called H^*-solution, detailed here [2]).

For some reasons, it is useful to introduce the following rescaled problem, using this change of variables:

$$v(t,x) = (1+\tau)^{\frac{1}{m-1}} u(\tau,x), \quad t = \log(1+\tau)$$

which transforms problem into the one we know.

$$\begin{cases} v_t + \mathcal{L}(v^m) = \frac{v}{m-1} & \text{in } (0,+\infty)\times\Omega, \\ v(0,x) = u_0(x) & \text{in } \Omega, \\ v(t,x) = 0 & \text{for } t>0 \text{ and } x \in \partial\Omega, \end{cases}$$

Then, we have the following results (see [2]):

Theorem

Let v be any nonnegative H^*-solution of the Problem , then we have (unless $v \equiv 0$)

$$\lim_{t\to\infty} \|v(t,\cdot) - S\|_{L^\infty(\Omega)} = 0,$$

Corollary

Let $m > 1$. For any given constant $c > 0$, there exists a unique weak solution to the elliptic equation

$$\mathcal{L}(S^m) = cS \quad \text{in } \Omega,$$

1.3 Usefull Results and Properties

For more clarity, we present some useful propreties and calculations that will be used along the work.

Properties on ∇, Δ and div

Here are some usefull properties on ∇, Δ and $\nabla\cdot = div$

$$\forall\,(A,B):\mathbf{R}^d\mapsto\mathbf{R}^d\ ,\quad \bullet\quad \Delta AB = B\Delta A + A\Delta B + 2\nabla A.\nabla B \tag{P1}$$

$$\bullet\quad A\Delta B + \nabla A.\nabla B = \nabla\cdot[A\nabla B] \tag{P2}$$

$$\bullet\quad A\Delta B + 2\nabla A.\nabla B = \frac{1}{A}div(A^2\nabla B) \tag{P3}$$

Integration by Part

Suppose Ω is an open bounded subset of $\mathbf{R}^d$ with a piecewise smooth boundary $\partial\Omega$, $\vec{dn}_{\partial\Omega}$ is the differential outward normal vector to the boundary. For $u:\Omega\mapsto\mathbf{R}$ *and* $\vec{V}:\Omega\mapsto\mathbf{R}^d$,

$$\int_\Omega u\,div(\vec{V})dx = \int_{\partial\Omega} u\,\vec{V}\cdot\,\vec{dn}_{\partial\Omega}\ - \int_\Omega \nabla u\cdot\vec{V}dx$$

GWPI: general weighted Poincaré inequality

Given m,Ω, there exists a constant $K>0$ such that for every stationary solution $S>0$ of the problem, and for every $\theta\in W_0^{1,2}(\Omega,S^{2m}\,dx)$ we have (see [4]):

$$\int_\Omega S_c^{2m}|\nabla\theta|^2\,dx \geq Kc\int_\Omega S_c^{m+1}|\theta-\bar{\theta}|^2\,dx. \tag{3.3}$$

Usefull Calculations

Here are some usefull asymptotical development, that are used later or could be used for the continuation of the research. Here, $\varphi \in \mathcal{C}^2(\mathbf{R})$, *bijective on* $\mathbf{R}_+$, $\varphi(0) = 0$, $\varphi' > 0$, $\varphi'' > 0$, $\theta, S > 0$, $\alpha = \varphi^{-1}(1) > 0$.

We define

$$K_\varphi(\theta, S) = \frac{\varphi'[\varphi^{-1}(\varphi(\theta+\alpha)\varphi(S))]}{\varphi(S)^2\varphi'(\theta+\alpha)} \quad \textit{and} \quad f_\varphi(\theta, S) = K_\varphi(\theta, S)\varphi(S)\left[\varphi^{-1}(\varphi(\theta+\alpha)\varphi(S)) - S\varphi(\theta+\alpha)\right]$$

Some long calculations give the following results:

- $\varphi(\theta+\alpha) \underset{\theta\to 0}{=} 1 + \ \theta\varphi'(\alpha) + \ \dfrac{\theta^2}{2}\varphi''(\alpha) + \ \mathcal{O}(\theta^3)$
- $\varphi^{-1}(\varphi(\theta+\alpha)\varphi(S)) \underset{\theta\to 0}{=} S + \ \theta\,\dfrac{\varphi(S)\varphi'(\alpha)}{\varphi'(S)} + \ \dfrac{\theta^2}{2}\,\dfrac{\varphi(S)}{\varphi'(S)}[\varphi''(\alpha) - \dfrac{\varphi(S)\varphi''(S)\varphi'(\alpha)^2}{\varphi'(S)^2}] + \ \mathcal{O}(\theta^3)$
- $K_\varphi(\theta, S) \underset{\theta\to 0}{=} \dfrac{\varphi'(S)}{\varphi(S)^2\varphi'(\alpha)} + \theta\dfrac{1}{\varphi(S)}[\dfrac{\varphi''(S)}{\varphi'(S)} - \dfrac{\varphi'(S)\varphi''(\alpha)}{\varphi(S)\varphi'(\alpha)^2}] + \mathcal{O}(\theta^2)$

- $f_\varphi(\theta, S) \underset{\theta\to 0}{=} \ \theta \cdot \dfrac{1}{\varphi(S)}(1 - S\dfrac{\varphi'(S)}{\varphi(S)})$
  $+ \ \dfrac{\theta^2}{2} \cdot \dfrac{1}{\varphi(S)}\ [\ \dfrac{\varphi''(\alpha)}{\varphi'(\alpha)}(1 - \dfrac{2}{\varphi'(\alpha)} - S\dfrac{\varphi'(S)}{\varphi(S)}(1 - \dfrac{2}{\varphi'(\alpha)})) + \dfrac{\varphi''(S)}{\varphi'(S)}((2 - \varphi'(\alpha))\dfrac{\varphi(S)}{\varphi'(S)} - 2S)\ ]$
  $+ \ \mathcal{O}(\theta^3)$

Chapter 2

A First Approach

2.1 Proof for the classic equation

In this section (see [4]) we will make some hypothesis to simplify the problem.

Hypothesis:

1. Let's fix the function φ as $\varphi = .^m$.
2. The fractionnal Laplacian is just a classic Laplacian: $-\mathcal{L} = \Delta$

So the PDE (3.1) becomes:

$$\begin{cases} \partial_t v = \Delta v^m + cv & in\ \mathbb{R}_+^* \times \Omega, \\ v(0,x) = u_0(x) & in\ \Omega, \\ v(t,x) = 0 & in\ \mathbb{R}_+^* \times\ \partial\Omega \end{cases} \tag{4.1}$$

Where $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary, $m > 1$ and $c > 0$.

We suppose S a stationary solution of (4.1), which means:

$$\begin{cases} \Delta S^m + cS = 0 & in\ \Omega, \\ \partial_t S = 0 & in\ \Omega \end{cases} \tag{4.2}$$

The goal is to evaluate the convergence rate of the solution v toward the stationnary state S. We need a good tool to measure this rate: we proceed with the following relative error.

$$\theta(t,x) = \frac{v(t,x)}{S(x)} - 1 \tag{4.3}$$

Let's proceed with the proof. First, we need to transform the PDE (4.1) to get an equation on θ.

PDE on the relative error

We have,

$$S\partial_t \frac{v}{S} = \Delta(\frac{v}{S}.S)^m + c\frac{v}{S}.S$$
$$S\partial_t \theta = \Delta[S^m(1+\theta)^m] + cS(1+\theta)$$

Using the formula (P1), we get:

$$S\partial_t\theta = (1+\theta)^m \Delta S^m + S^m \Delta (1+\theta)^m + 2\nabla S^m \cdot \nabla (1+\theta)^m + cS(1+\theta)$$

Using the formula (P2), we get:

$$S\partial_t\theta = \nabla \cdot [S^m \nabla (1+\theta)^m] + (1+\theta)^m \Delta S^m + \nabla S^m . \nabla (1+\theta)^m + cS(1+\theta)$$

Using the stationnary property (4.2),

$$S\partial_t\theta = \nabla \cdot [S^m \nabla (1+\theta)^m] - cS.(1+\theta)^m + \nabla S^m . \nabla (1+\theta)^m + cS(1+\theta)$$

Let $f(\theta) = 1 + \theta - (1+\theta)^m$,

$$S\partial_t\theta = \nabla \cdot [S^m \nabla (1+\theta)^m] + \nabla S^m . \nabla (1+\theta)^m + cSf(\theta)$$

$$S^{m+1}\partial_t\theta = S^m \nabla \cdot [S^m \nabla (1+\theta)^m] + S^m \nabla S^m . \nabla (1+\theta)^m + cS^{m+1} f(\theta)$$

$$S^{m+1}\partial_t\theta = \nabla \cdot [S^{2m} \nabla (1+\theta)^m] + cS^{m+1} f(\theta)$$

Finally,

$$\partial_t\theta = \frac{1}{S^{m+1}} \nabla \cdot [S^{2m} \nabla (1+\theta)^m] + cf(\theta) \tag{4.4}$$

We are now going to prove a certain type of entropy inequalities.

Entropy function

We define the entropy function :

$$E[\theta(t)] := \frac{1}{2} \int_\Omega \left|\theta(t,x) - \bar{\theta}\right|^2 S^{1+m}\, dx \tag{4.5}$$

where

$$\bar{\theta}(t) := \frac{\int_\Omega \theta(t,x) S^{1+m}\, dx}{\int_\Omega S^{1+m}\, dx} = \text{relative error to this sheaf}$$

and S is the chosen solution to the elliptic problem:

$$\begin{cases} -\Delta S^m = cS & \text{in } \Omega, \\ S > 0 & \text{in } \Omega, \\ S = 0 & \text{on } \partial\Omega. \end{cases}$$

We recall that the goal here will be to show the rate of convergence of the entropy function $E[\theta(t)]$ to 0 as $t \to 0$. To do so, we want to get an differential inequation on $E[\theta(t)]$ and then use the Gronwall Lemma on it.

Differentiation of the Entropy function

Uniform convergence in relative error holds as $t \to \infty$, that means $\epsilon(t) := \|\theta(t,\cdot)\|_\infty \to 0$.

First we notice that

$$\int_\Omega [\theta(t,x) - \bar{\theta}(t)]\, \partial_t\bar{\theta}\ S^{m+1} dx = \partial_t\bar{\theta}(t) \int_\Omega [\theta(t,x) - \bar{\theta}(t)]\ S^{m+1} dx = 0,$$

and also that

$$0 \leq \int_\Omega [\theta(t,x) - \bar{\theta}(t)]^2\ S^{m+1} dx = \int_\Omega [\theta(t,x)^2 - \bar{\theta}(t)^2]\ S^{m+1} dx$$

We next differentiate $E[\theta(t)]$,

$$\begin{aligned}\frac{d}{dt}E[\theta(t)] &= \int_\Omega [\theta(t)-\bar\theta]\ \partial_t\theta S^{m+1}\,dx - \int_\Omega [\theta(t)-\bar\theta]\ \partial_t\bar\theta S^{m+1}\,dx \\ &= \int_\Omega [\theta(t)-\bar\theta]\ [S^{m+1}\partial_t\theta]\,dx \\ &= \int_\Omega [\theta(t)-\bar\theta]\ (\nabla\cdot[S^{2m}\nabla(1+\theta)^m] + cS^{m+1}f(\theta))\,dx \\ &= \int_\Omega [\theta(t)-\bar\theta]\ \nabla\cdot[S^{2m}\nabla(1+\theta)^m]dx + c\int_\Omega [\theta(t)-\bar\theta]f(\theta)\,S^{m+1}dx\end{aligned}$$

Using integration by parts,

$$\int_\Omega [\theta(t)-\bar\theta]\ \nabla\cdot[S^{2m}\nabla(1+\theta)^m]dx = \underbrace{\int_{\partial\Omega}[\theta(t)-\bar\theta].S^{2m}.\nabla(1+\theta)^m\,\vec{dn}_{\partial\Omega}}_{=0\ because\ S=0\ on\ \partial\Omega} - \int_\Omega \nabla[\theta(t)-\bar\theta]\cdot[S^{2m}\nabla(1+\theta)^m]dx$$

So,

$$\frac{d}{dt}E[\theta(t)] = \underbrace{-m\int_\Omega (1+\theta)^{m-1}\|\nabla\theta\|^2 S^{2m}dx}_{=\ (I)} + \underbrace{c\int_\Omega [\theta(t)-\bar\theta]f(\theta)\,S^{m+1}dx}_{=\ (II)}$$

Differential inequation

We also notice that for small $\theta-\bar\theta$ and we have:

$$f(\theta) = f(\bar\theta) + f'(\bar\theta)(\theta-\bar\theta) + \frac{1}{2}f''(\tilde\theta)(\theta-\bar\theta)^2, \quad \text{with Rolle theorem}$$

where $\tilde\theta$ lies between θ and $\bar\theta$. We then have:

$$\begin{aligned}\frac{(II)}{c} &= \int_\Omega [f(\theta)-f(\bar\theta)][\theta-\bar\theta]S^{m+1}\,dx + \underbrace{\int_\Omega f(\bar\theta)[\theta-\bar\theta]\,S^{m+1}dx}_{=\ f(\bar\theta)\int_\Omega[\theta-\bar\theta]\,S^{m+1}dx\ =\ 0} \\ &= f'(\bar\theta)\int_\Omega [\theta-\bar\theta]^2\,S^{m+1}dx + \frac{1}{2}\int_\Omega f''(\tilde\theta)[\theta-\bar\theta]^3\,S^{m+1}dx \\ &= f'(\bar\theta)\ 2E[\theta(t)] + \frac{1}{2}\int_\Omega f''(\tilde\theta)[\theta-\bar\theta]^3\,S^{m+1}dx\end{aligned}$$

Now we use the fact that

$$f'(\bar\theta) \underset{t\to+\infty}{=} 1-m+m(1-m)\bar\theta + o(\bar\theta(t)) \ < 0 \text{ for t big enough}$$

$$f''(\tilde\theta) \underset{t\to+\infty}{\to} m(1-m) \ < 0$$

For t big enough,

$$\begin{aligned}
-\frac{(II)}{c} &= -f'(\bar{\theta})\ 2E[\theta(t)] - \frac{1}{2}\int_\Omega f''(\tilde{\theta})[\theta-\bar{\theta}]^3\, S^{m+1}dx \\
&\geq 2(m-1)\ E[\theta(t)] - \frac{1}{2}\int_\Omega f''(\tilde{\theta})[\theta-\bar{\theta}]^3\, S^{m+1}dx \\
&\geq 2(m-1)\ E[\theta(t)] + \frac{1}{2}\int_\Omega \underbrace{[-f''(\tilde{\theta})]}_{>0\ for\ t\ big}\ |\theta-\bar{\theta}|^2\ \underbrace{(\theta-\bar{\theta})}_{\geq -2\epsilon(t)} S^{m+1}dx \\
&\geq 2(m-1)\ E[\theta(t)] - \epsilon(t)\int_\Omega \underbrace{[-f''(\tilde{\theta})]}_{\leq m(m-1)+\delta,\ \delta>0}\ |\theta-\bar{\theta}|^2\ S^{m+1}dx \\
&\geq 2(m-1)\ E[\theta(t)] - \epsilon(t)\ [m(m-1)+\delta]\ 2E[\theta(t)] \\
&\geq -2[\ 1-m+(m(m-1)+\delta)\ \epsilon(t)\ ]\ E[\theta(t)]
\end{aligned}$$

So by minoring the second term (II), we finally get:

$$-\frac{d}{dt}E[\theta(t)] \geq m[1+\epsilon(t)]^{m-1}\int_\Omega |\nabla\theta|^2 S^{2m}\,dx - 2c[\ 1-m+(m(m-1)+\delta)\ \epsilon(t)\ ]\ E[\theta(t)]$$

Now, we need a Poincaré inequatlity for minoring the first term (I).

Using the Poincaré inequality (3.3), there exist $K>0$ such that

$$\int_\Omega S^{2m}|\nabla\theta|^2\,dx \geq Kc\int_\Omega S^{m+1}|\theta-\bar{\theta}|^2\,dx.$$

Main assumption: We now make an assumption:

$$Km - 2(1-m) \geq \lambda_0 > 0.$$

Putting these things together, we have obtained the desired differential inequation:

$$-\frac{d}{dt}E[\theta(t)] \geq c\ (\ Km[1-\epsilon(t)]^{m-1} - 2[1-m+(m(m-1)+\delta)\epsilon(t)]\ )\ E[\theta(t)] \tag{4.6}$$

Gronwall's Lemma

Finally, let's apply Gronwall's lemma:

$$\begin{aligned}
-\frac{d}{dt}E[\theta(t)] &\geq c\ (\ Km[1-\epsilon(t)]^{m-1} + 2(m-1) - 2m(m-1)\epsilon(t) - 2\delta\epsilon(t)\ )\ E[\theta(t)] \\
&\geq c\ (\ Km + 2(m-1) - [(K+2)m(m-1)+2\delta]\epsilon(t) + o(\epsilon(t))\ )\ E[\theta(t)],\ as\ t\to+\infty \\
&\geq c\ (\ Km + 2(m-1) - \delta')\ E[\theta(t)]\ \text{ for any }\ \delta'>0 \text{ and t large enough}
\end{aligned}$$

Then, with Gronwall,

$$\forall\beta < c(\ Km+2(m-1)),\ \exists\ t^*_{m,d,\beta},\ K>0|\ \forall t\leq t^*,\quad \boxed{E[\theta(t)] \leq E[\theta(t^*)]e^{-\beta(t-t^*)}} \tag{4.7}$$

This is the desired rate of convergence that controls the relative error. □

2.2 Proof for the generalised equation: an attempt

In this section, we do not fix the function φ anymore. The goal here is to generalise the previous proof (see [3]).

Hypothesis:

1. $\varphi \in \mathcal{C}^2(\mathbb{R})$, bijective, $\varphi' > 0$, $\varphi(\pm\infty) = \pm\infty$ and $\varphi(0) = 0$
2. The fractionnal Laplacian is a classic Laplacian: $-\mathcal{L} = \Delta$

So the PDE (3.1) becomes:

$$\begin{cases} \partial_t v = \Delta\varphi(v) + cv & in\ \mathbb{R}_+^* \times \Omega, \\ v(0,x) = u_0(x) & in\ \Omega, \\ v(t,x) = 0 & in\ \mathbb{R}_+^* \times\ \partial\Omega \end{cases} \tag{4.8}$$

Where $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary and $c > 0$.
We suppose S a stationary solution of (4.8), which means:

$$\begin{cases} \Delta\varphi(S) + cS = 0 & in\ \Omega, \\ \partial_t S = 0 & in\ \Omega \end{cases} \tag{4.9}$$

The goal is to evaluate the convergence rate of the solution v toward the stationnary state S. To quantify the convergence, we will use a relative error. The relative error formula is different from the previous one. Otherwise, it would lead to impossible calculations. For more commodities, let's choose the Generalised Relative Error:

$$\theta(x,t) = \varphi^{-1}\left(\frac{\varphi(v(x,t))}{\varphi(S(x))}\right) - \alpha, \quad with\ \alpha = \varphi^{-1}(1) \tag{4.10}$$

Remark: The Generalised Relative Error coincide with the standard one when $\varphi = .^m$:

$$\varphi^{-1}\left(\frac{\varphi(v)}{\varphi(S)}\right) - \alpha = \left(\frac{v^m}{S^m}\right)^{1/m} - 1^{1/m} = \frac{v}{S} - 1$$

Let's proceed with the proof. Like the previous section, we need to transform the PDE (4.8) to get an equation on θ.

PDE on the Generalised Relative Error

First,

$$\theta = \varphi^{-1}\left(\frac{\varphi(v)}{\varphi(S)}\right) - \alpha \ \Leftrightarrow\ \varphi(\theta+\alpha) = \frac{\varphi(v)}{\varphi(S)} \ \Leftrightarrow\ \varphi(\theta+\alpha)\varphi(S) = \varphi(v)$$

So, the partial differential equation (4.8) becomes

$$\partial_t v = \Delta[\varphi(\theta+\alpha)\varphi(S)] + cv$$

Using the formula (P1), we get:

$$\partial_t v = \varphi(S)\Delta\varphi(\theta+\alpha) + \varphi(\theta+\alpha)\Delta\varphi(S) + 2\nabla\varphi(S)\cdot\nabla\varphi(\theta+\alpha) + cv$$

Using the stationary property,

$$\partial_t v = \varphi(S)\Delta\varphi(\theta+\alpha) - cS\varphi(\theta+\alpha) + 2\nabla\varphi(S)\cdot\nabla\varphi(\theta+\alpha) + cv$$

Using the formula (P3),

$$A\Delta B + 2\nabla A.\nabla B = \frac{1}{A} div(A^2\nabla B)$$

We get:

$$\partial_t v = \frac{1}{\varphi(S)} div(\,\varphi(S)^2\nabla\varphi(\theta+\alpha)\,) + c\,[v - S\varphi(\theta+\alpha)]$$

We should have an expression uniquely function of θ and S, so let's recall that

$$v = \varphi^{-1}[\varphi(\theta+\alpha)\varphi(S)]$$

Then, the equation becomes,

$$\frac{\varphi(S)^2\varphi'(\theta+\alpha)}{\varphi'[\varphi^{-1}(\varphi(\theta+\alpha)\varphi(S))]}\partial_t\theta = div(\,\varphi(S)^2\nabla\varphi(\theta+\alpha)\,) + c\,\varphi(S)[\varphi^{-1}(\varphi(\theta+\alpha)\varphi(S)) - S\varphi(\theta+\alpha)]$$

For more clarity, let's define

$$K_\varphi(\theta,S) := \frac{\varphi'[\varphi^{-1}(\varphi(\theta+\alpha)\varphi(S))]}{\varphi(S)^2\varphi'(\theta+\alpha)} \quad and \quad f_\varphi(\theta,S) := K_\varphi(\theta,S)\varphi(S)[\varphi^{-1}(\varphi(\theta+\alpha)\varphi(S)) - S\varphi(\theta+\alpha)]$$

then

$$\boxed{\partial_t\theta = K_\varphi(\theta,S)\,div(\,\varphi(S)^2\nabla\varphi(\theta+\alpha)\,) + c\,f_\varphi(\theta,S)} \tag{4.11}$$

Remark: This equation is exactly the generalisation of the one from the previous section:

$$"\partial_t\theta = \frac{1}{S^{m+1}}\nabla\cdot[S^{2m}\nabla(1+\theta)^m] + cf(\theta)"$$

Indeed,

$$K_{x\mapsto x^m}(\theta,S) = \frac{1}{S^{m+1}} \quad f_{x\mapsto x^m}(\theta,S) = f(\theta)$$

Entropy function and Weight

We define the entropy function :

$$E[\theta(t)] = \frac{1}{2}\int_\Omega \left|\theta(t,x) - \bar\theta\right|^2 A_\varphi(S)\,dx \tag{4.12}$$

where we call $A_\varphi(S) > 0$ an unknown weight we should fix later. $A_\varphi(S)$ has to respect $A_{x\mapsto x^m}(S) = S^{m+1}$, in order to fit the previous case.
We set $d\mu = A_\varphi(S(x))dx$ for more simplicity.

$$\bar\theta(t) = \frac{\int_\Omega \theta(t,x)A_\varphi(S)\,dx}{\int_\Omega A_\varphi(S)\,dx} = \text{relative error to this sheaf}$$

and S is the chosen solution to the elliptic problem:

$$\begin{cases} -\Delta\varphi(S) = cS & \text{in } \Omega, \\ S > 0 & \text{in } \Omega, \\ S = 0 & \text{on } \partial\Omega. \end{cases}$$

We recall that the goal here will be to show the rate of convergence of the entropy function $E[\theta(t)]$ to 0 as $t \to 0$. To do so, we want to get a differential inequation on $E[\theta(t)]$ and then use the Gronwall Lemma on it.

Differentiation of entropy function

Uniform convergence in relative error holds as $t \to \infty$, that means $\epsilon(t) = \|\theta(t,\cdot)\|_\infty \to 0$.

First we notice that

$$\int_\Omega [\theta(t,x) - \bar\theta(t)]\, \partial_t\bar\theta \; d\mu = \partial_t\bar\theta(t) \int_\Omega [\theta(t,x) - \bar\theta(t)] \; d\mu = 0,$$

and also that

$$0 \le \int_\Omega [\theta(t,x) - \bar\theta(t)]^2 \; d\mu = \int_\Omega [\theta(t,x)^2 - \bar\theta(t)^2] \; d\mu$$

We next differentiate $E[\theta(t)]$,

$$\begin{aligned}\frac{d}{dt}E[\theta(t)] &= \int_\Omega [\theta(t) - \bar\theta] \; \partial_t\theta d\mu - \underbrace{\int_\Omega [\theta(t) - \bar\theta] \; \partial_t\bar\theta d\mu}_{=0} \\ &= \underbrace{\int_\Omega [\theta(t) - \bar\theta] \; K_\varphi(\theta,S) \; div(\,\varphi(S)^2 \nabla\varphi(\theta+\alpha)\,) A_\varphi(S) dx}_{=(I)} + c \underbrace{\int_\Omega [\theta(t) - \bar\theta] f_\varphi(\theta,S)\, d\mu}_{=(II)}\end{aligned}$$

Using Integration By Parts,

$$(I) = \underbrace{\int_{\partial\Omega} [\theta(t) - \bar\theta] \; K_\varphi(\theta,S)\, \varphi(S)^2 A_\varphi(S)\, \vec\nabla\varphi(\theta+\alpha) \cdot \vec{dn}_{\partial\Omega}}_{=0 \; because \; S=0 \; on \; \partial\Omega} - \int_\Omega \nabla[\, K_\varphi(\theta,S) A_\varphi(S)(\theta(t) - \bar\theta)] \cdot [\varphi(S)^2 \nabla\varphi(\theta+\alpha)] dx$$

Now, it is tempting to set $A_\varphi(S) \overset{?}{=} \frac{1}{K_\varphi(\theta,S)}$, for 2 reasons: the first one is that it simplifies the calculation in the nabla $\nabla[...]$. The second one is that it would be the perfect generalisation of the previous section since that:

$$\frac{1}{K_{\varphi = x \mapsto x^m}(\theta,S)} = S^{m+1}$$

However, it is impossible due to the fact that the weight $A_\varphi(S)$ cannot be θ-dependant !

For the moment let's keep the general notation $A_\varphi(S)$. Now, we continue the calculation using the general weight.

Calculations with general weight

$$\frac{d}{dt}E[\theta(t)] = (I) + c\,(II) \tag{4.13}$$

First we focus on (I):

$$\begin{aligned}(I) &= -\int_\Omega \nabla[\, K_\varphi(\theta,S) A_\varphi(S)(\theta(t) - \bar\theta)] \cdot [\varphi(S)^2 \nabla\varphi(\theta+\alpha)] dx \\ &= \underbrace{-\int_\Omega \|\nabla\theta\|^2 \, \varphi'(v)\, A_\varphi(S) \; dx}_{(I)_1} \underbrace{- \int_\Omega [\theta(t) - \bar\theta]\, K_\varphi(\theta,S)\, \varphi(S)^2\, \varphi'(\theta+\alpha)\, \nabla A_\varphi(S) \cdot \nabla\theta \; dx}_{(I)_2} \\ &\qquad \underbrace{- \int_\Omega [\theta(t) - \bar\theta]\, A_\varphi(S)\, \varphi(S)^2\, \varphi'(\theta+\alpha)\, \nabla K_\varphi(\theta,S) \cdot \nabla\theta \; dx}_{(I)_3}\end{aligned}$$

First,

$$(I)_2 = -\int_\Omega [\theta(t) - \bar{\theta}]\, A'_\varphi(S)\, \varphi'(v)\, \nabla S \cdot \nabla\theta \; dx$$

Now, let's calculate the third term. For that let's find the expression of $\nabla K_\varphi(\theta, S)$
After derivation of the expression, we find:

$$\nabla K_\varphi(\theta, S) = \frac{\varphi(S)^2\varphi'(\theta+\alpha)\varphi''(v)\nabla v - 2\varphi'(v)\varphi'(\theta+\alpha)\varphi(S)\varphi'(S)\nabla S - \varphi'(v)\varphi(S)^2\varphi''(\theta+\alpha)\nabla\theta}{\varphi(S)^4\varphi'(\theta+\alpha)^2}$$

with

$$\nabla v = \frac{\varphi'(S)\varphi(\theta+\alpha)\nabla S + \varphi(S)\varphi'(\theta+\alpha)\nabla\theta}{\varphi'(v)}$$

We find the expression of $\varphi(S)^2\, \varphi'(\theta+\alpha)\, \nabla K_\varphi(\theta, S) \cdot \nabla\theta$:

$$\begin{aligned}\varphi(S)^2\, \varphi'(\theta+\alpha)\, \nabla K_\varphi(\theta, S) \cdot \nabla\theta = \nabla S \cdot \nabla\theta\, [\frac{\varphi''(v)\varphi(\theta+\alpha)}{\varphi'(v)} - 2\frac{\varphi'(v)}{\varphi(S)}]\, \varphi'(S) \\ + \|\nabla\theta\|^2\, [\frac{\varphi''(v)\varphi'(\theta+\alpha)\varphi(S)}{\varphi'(v)} - \frac{\varphi'(v)\varphi''(\theta+\alpha)}{\varphi'(\theta+\alpha)}]\end{aligned}$$

Then,

$$\begin{aligned}(I)_3 = &-\int_\Omega [\theta(t) - \bar{\theta}]\, A_\varphi(S)\; \nabla S \cdot \nabla\theta\, [\frac{\varphi''(v)\varphi(\theta+\alpha)}{\varphi'(v)} - 2\frac{\varphi'(v)}{\varphi(S)}]\, \varphi'(S)\; dx \\ &-\int_\Omega [\theta(t) - \bar{\theta}]\, A_\varphi(S)\; \|\nabla\theta\|^2\, [\frac{\varphi'(v)\varphi'(\theta+\alpha)\varphi(S)}{\varphi'(v)} - \frac{\varphi'(v)\varphi''(\theta+\alpha)}{\varphi'(\theta+\alpha)}]\; dx\end{aligned}$$

Finally, we get the final general expression for any $A_\varphi(S)$:

$$\frac{d}{dt}E[\theta(t)] = \tag{4.14}$$

$$\text{terms involving } \|\nabla\theta\|^2 \quad \begin{cases} -\displaystyle\int_\Omega \|\nabla\theta\|^2\, \varphi'(v)\, A_\varphi(S)\, dx \\ -\displaystyle\int_\Omega [\theta(t) - \bar{\theta}]\; \|\nabla\theta\|^2\, [\frac{\varphi''(v)\varphi'(\theta+\alpha)\varphi(S)}{\varphi'(v)} - \frac{\varphi'(v)\varphi''(\theta+\alpha)}{\varphi'(\theta+\alpha)}]\; A_\varphi(S)\, dx \end{cases}$$

$$\text{terms involving } \nabla S \cdot \nabla\theta \quad \begin{cases} -\displaystyle\int_\Omega [\theta(t) - \bar{\theta}]\, \nabla S \cdot \nabla\theta\, [A_\varphi(S)\frac{\varphi'(S)\varphi''(v)\varphi(\theta+\alpha)}{\varphi'(v)} - 2A_\varphi(S)\frac{\varphi'(S)\varphi'(v)}{\varphi(S)} + A'_\varphi(S)\varphi'(v)]dx \\ \quad + c\displaystyle\int_\Omega [\theta(t) - \bar{\theta}] f_\varphi(\theta, S)\, A_\varphi(S)\; dx \end{cases}$$

Choice of the entropic Weight

Let's recall that the goal is to use the Gronwall lemma. This means that we want to minor $-\frac{d}{dt}E[\theta(t)]$ with something positive times $E[\theta(t)]$,

$$i.e. -\frac{d}{dt}E[\theta(t)] \geq [smth\ positive] \cdot E[\theta(t)]$$

We need first to simplify the expression of $\frac{d}{dt}E[\theta(t)]$, with the choice of a good weight. Let's recall that the weight $A_\varphi(S)$ must be such that:

1. $A_\varphi(S) > 0$
2. $A_{x\mapsto x^m}(S) = S^{m+1}$

There are different possible candidates, like $A_\varphi(S) = S\varphi(S)$ which is the most obvious one. However, this calculations does not seems to lead anywhere (I tried).

Another candidate is the following. Previously, I said that it was tempting to set $A_\varphi(S) \stackrel{?}{=} \frac{1}{K_\varphi(\theta,S)}$ because it cancels some terms in the nabla $\nabla[...]$, but it was not possible because $A_\varphi(S)$ cannot be $\theta - dependant$. Keeping the same idea, I will suggest to take the limit

$$\boxed{A_\varphi(S) = \lim_{\theta\to+0} \frac{1}{K_\varphi(\theta,S)} = \frac{\varphi'(\alpha)\varphi(S)^2}{\varphi'(S)}} \qquad \text{which indeed satisfies the 2 conditions.} \tag{4.15}$$

That choice should simply the calculations in the nabla $\nabla[...]$ asymptoticlally... To verify if it is the case, let's replace $A_\varphi(S)$ in the following expression:

$$\begin{aligned}
&[A_\varphi(S)\frac{\varphi'(S)\varphi''(v)\varphi(\theta+\alpha)}{\varphi'(v)} - 2A_\varphi(S)\frac{\varphi'(S)\varphi'(v)}{\varphi(S)} + A'_\varphi(S)\varphi'(v)] \\
= \;& \frac{\varphi'(\alpha)\varphi(S)^2\varphi''(v)\varphi(\theta+\alpha)}{\varphi'(v)} - 2\varphi'(\alpha)\varphi(S)\varphi'(v) + 2\varphi(S)\varphi'(v)\varphi'(\alpha) - \frac{\varphi'(\alpha)\varphi'(v)\varphi''(S)\varphi(S)^2}{\varphi'(S)^2} \\
\underset{t\to+\infty}{\to}\; & \frac{\varphi'(\alpha)\varphi(S)^2\varphi''(S)}{\varphi'(S)} - 2\varphi'(\alpha)\varphi(S)\varphi'(S) + 2\varphi(S)\varphi'(S)\varphi'(\alpha) - \frac{\varphi'(\alpha)\varphi'(S)\varphi''(S)\varphi(S)^2}{\varphi'(S)^2} = 0
\end{aligned}$$

So this choice seems to be a smart one. Now, let's replace $A_\varphi(S)$ in the expression. We get:

$$\begin{aligned}
&\frac{d}{dt}E[\theta(t)] = \qquad\qquad (4.16)\\
&-\int_\Omega \|\nabla\theta\|^2\,\varphi'(v)\,d\mu \\
&-\int_\Omega [\theta(t)-\bar\theta]\;\|\nabla\theta\|^2\,[\frac{\varphi''(v)\varphi'(\theta+\alpha)\varphi(S)}{\varphi'(v)} - \frac{\varphi'(v)\varphi''(\theta+\alpha)}{\varphi'(\theta+\alpha)}]\,d\mu \\
&-\int_\Omega [\theta(t)-\bar\theta]\;\nabla S\cdot\nabla\theta\;\underbrace{[\frac{\varphi'(\alpha)\varphi(S)^2\varphi''(v)\varphi(\theta+\alpha)}{\varphi'(v)} - 2\varphi'(\alpha)\varphi(S)\varphi'(v) + 2\varphi(S)\varphi'(v)\varphi'(\alpha) - \frac{\varphi'(\alpha)\varphi'(v)\varphi''(S)\varphi(S)^2}{\varphi'(S)^2}]}_{\underset{t\to+\infty}{\to}0}\;dx \\
&+\;c\int_\Omega [\theta(t)-\bar\theta] f_\varphi(\theta,S)\,d\mu
\end{aligned}$$

2.3 Paths to explore

Now, the goal is to find an inequality of this type in order to apply Gronwall Lemma:

$$i.e. -\frac{d}{dt}E[\theta(t)] \geq [smth\ positive] \cdot E[\theta(t)]$$

However, I did not had the time in my internship to continue.

There are many path to explore in order to continue the proof:

- Making further assumption on φ and its derivatives in order to be able to minor each integral the way we want. For exemple, the assumption could be to make φ *close* to a power function(see [3]).

- Studying the asymptotic behaviour of the third integral could be useful, maybe to show that it is neglectable in regards of the others integrals...

- Adapting the general weighted Poincaré inequality to the generalised equation should also help to bound the integrals(see [4]).

- ...

Chapter 3

An other Approach to the Problem

An other approach to the problem is to focus on the corresponding linearized equation.

3.1 Linearized problem with classic Laplacian

In this section we will make some hypothesis to simplify the problem.

Hypothesis:

1. $\varphi \in \mathcal{C}^2(\mathbb{R})$, bijective, $\varphi' > 0$, $\varphi(\pm\infty) = \pm\infty$ and $\varphi(0) = 0$
2. The fractionnal Laplacian is a classic Laplacian: $-\mathcal{L} = \Delta$

We note $\psi = \varphi^{-1}$ for more commodities, and we make the next variable change:

$$\partial_t u = \Delta\varphi(u) + c\,u \;\Leftrightarrow\; \partial_t \psi(v) = \Delta v + c\,\psi(v)$$

So the PDE (3.1) becomes:

$$\begin{cases} \partial_t \psi(v) = \Delta v + c\,\psi(v) & in\ \mathbb{R}_+^* \times \Omega, \\ v(0,x) = u_0(x) & in\ \Omega, \\ v(t,x) = 0 & in\ \mathbb{R}_+^* \times\ \partial\Omega \end{cases} \tag{5.1}$$

Now, S represent the stationnary solution of (5.1):

$$\begin{cases} \Delta S = -c\psi(S) & in\ \Omega, \\ \partial_t S = 0 & in\ \Omega \end{cases} \tag{5.2}$$

and $v(x,t) \xrightarrow[t\to+\infty]{L^\infty} S(x)$
For now, we want to linearize the PDE to simplify the problem (see [5]).

Linearization

Let $\mathcal{H}$, the operator that symbolize the equation (5.1):

$$\mathcal{H} := \partial_t \psi(.) - \Delta. + c\,\psi(.)$$

(5.1) becomes $\mathcal{H}v = 0$. We also have $\mathcal{H}S = 0$

We linearize the operator $\mathcal{H}$ around the stationnary solution S: $v := S + w$

$$\mathcal{H}v = \mathcal{H}(S+w) = \partial_t\psi(S+w) - \Delta(S+w) - c\psi(S+w)$$

$$\psi(S+w) \underset{w\to 0}{=} \psi(S) + w\,\psi'(S) + w^2\,\epsilon(w), \quad \text{with } \epsilon(w) \text{ bounded.}$$

Then we find,

$$\mathcal{H}v \underset{w\to 0}{=} \mathcal{H}S + \psi'(S)\,\partial_t w - \Delta(w) - c\,\psi'(S)\,w + \ldots \textit{ non linear terms}$$

Cancelling the non-linerar terms, the linearized equation is :

$$\boxed{\partial_t w = c\,w + \frac{\Delta w}{\psi'(S)}} \quad (5.3)$$

New entropy function

Now let us define the linear Entropy functional $\mathcal{E}[w]$ and the linear Fisher information $\mathcal{I}[w]$ as follows:

$$\mathcal{E}[w] = \frac{1}{\psi'(1)}\int_\Omega w^2\,\psi'(S)\,dx, \quad \mathcal{I}[w] = \int_\Omega |\nabla w|^2\,dx - \psi'(1)\,c\,\mathcal{E}[w]$$

Next, $\mathcal{E}[w]$ need to be differentiated in order to apply Gronwall, like before.

Differentiation

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}[w(t)] &= \frac{2}{\psi'(1)}\int_\Omega w\,\partial_t w\,\psi'(S)\,dx \\
&= \frac{2}{\psi'(1)}\int_\Omega w\,(\Delta w + c\,\psi'(S)\,w)\,dx \quad \text{replacing with (5.3)} \\
&= \frac{2}{\psi'(1)}\int_\Omega w\,\Delta w\,dx + 2c\,\mathcal{E}[w] \\
&= -\frac{2}{\psi'(1)}\int_\Omega |\nabla w|^2\,dx + 2c\,\mathcal{E}[w], \quad \text{using Intgration By Part.}
\end{aligned}$$

$$\Rightarrow \frac{d}{dt}\mathcal{E}[w] = -\frac{2}{\psi'(1)}\mathcal{I}[w] \quad (5.4)$$

Gronwall's Lemma

Now, we make the assumption of Poincaré Hypothesis (See section 3.2). This Poincaré inequality has been already used in [4] for the case $\psi = .^p$.
We will assume that for any "good" ψ , $\exists\lambda_\psi > 0 \mid \lambda_\psi\,\mathcal{E}[w] \le \mathcal{I}[w]$.

$$\Rightarrow \frac{d}{dt}\mathcal{E}[w] \le -\frac{2\lambda_\psi}{\psi'(1)}\mathcal{E}[w](t)$$

With Gronwall lemma,

$$\boxed{\mathcal{E}[w(t)] \le e^{-\frac{2\lambda_\psi}{\psi'(1)}t}\,\mathcal{E}[w_0]} \quad (5.5)$$

□

3.2 Linearized problem with fractionnal Laplacian

In this section, we will attempt the same type of proof, but with the fractionnal Laplacian and some other hypthesis.

Hypothesis:

1. Specific case of $\varphi = .^m$
2. The fractionnal Laplacian is set to the fractionnal power $1/2$: $-\mathcal{L} = -(-\Delta)^{1/2}$

We note $p = \frac{1}{m}$ for more commodities, and we make the next variable change like in the previou section. So the PDE (3.1) becomes:

$$\begin{cases} \partial_t v^p = -(-\Delta)^{1/2} v + c\, v^p & in\ \mathbb{R}_+^* \times \Omega, \\ v(0,x) = u_0(x) & in\ \Omega, \\ v(t,x) = 0 & in\ \mathbb{R}_+^* \times\ \partial\Omega \end{cases} \tag{5.6}$$

Now, S represent the stationnary solution of (5.6):

$$\begin{cases} (-\Delta)^{1/2} S = c\, S^p & in\ \Omega, \\ \partial_t S = 0 & in\ \Omega \end{cases} \tag{5.7}$$

and $v(x,t) \xrightarrow[t\to+\infty]{L^\infty} S(x)$

Linearization

The linearized equation in that case is the folowing (same process than the previous section):

$$\boxed{\partial_t w = c\, w - \frac{\mathcal{L}(w)}{p S^{p-1}}} \tag{5.8}$$

where $\mathcal{L}(.) = (-\Delta)^{1/2}(.)$

Entropy and differentiation

$$\mathcal{E}[w] = \int_\Omega w^2\, S^{p-1}\, dx, \quad \mathcal{I}[w] = \int_\Omega |\nabla w|^2\, dx - p\, c\, \mathcal{E}[w]$$

The case is similar to the previous one up to the Integration by Bart step!

$$\frac{d}{dt}\mathcal{E}[w(t)] = ... = -\frac{2}{p}\int_\Omega w\, \mathcal{L}(w)\, dx + 2c\, \mathcal{E}[w] = ?$$

So we need to estimate this integral:

$$\int_\Omega w\, \mathcal{L}(w)\, dx \quad = \quad ?$$

Like in [3], we extend the function $w(t,x)$ to a function $\hat{w}(t,x,y)$ defined for (x,y) in $\Omega\times\mathbb{R}_+$, such that:

$$\begin{cases} \hat{w}=0 & \text{on } \Omega^c\times\{y=0\},\\ \Delta\hat{w}=0 & \text{in } \mathbb{R}_+\times\Omega\times\mathbb{R}_+,\\ \partial_t w = c\,w - \frac{\partial_y\hat{w}}{pS^{p-1}} & \text{in } \mathbb{R}_+\times\Omega. \end{cases} \tag{5.9}$$

We have the following property (see [2]):

$$\partial_y\hat{w}(t,x,y)_{|y=0} = -\mathcal{L}(w)(t,x) \tag{5.10}$$

Then,

$$\begin{aligned} \int_\Omega w\,\mathcal{L}(w)\,dx &= -\int_\Omega \hat{w}(t,x,y)_{|y=0}\ \partial_y\hat{w}(t,x,y)_{|y=0}\ dx\\ &= \int_{\Omega\times\mathbb{R}^+}\hat{w}(t,x,y)\ \Delta\hat{w}\,dx\,dy\ +\ \int_{\Omega\times\mathbb{R}^+}\|\nabla\hat{w}\|^2\,dx\,dy\\ &= \int_{\Omega\times\mathbb{R}^+}\|\nabla\hat{w}\|^2\,dx\,dy \end{aligned} \tag{5.11}$$

$$\frac{d}{dt}\mathcal{E}[w(t)] = -\frac{2}{p}\int_{\Omega\times\mathbb{R}^+}\|\nabla\hat{w}\|^2\,dx\,dy + 2c\,\mathcal{E}[w]$$

In order to use Gronwall Lemma, a way to link (5.11) with $\mathcal{I}[w]$ must be found.

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